Dirac Delta-Function (Distribution)

Defining property

$$\int_{x_0}^{x_1} \mathrm{f}(x)\delta(x-a)\mathrm{d}x = \mathrm{f}(a) \qquad \text{for } x_0 < a < x_1,\ f \in \mathcal{C}$$

Useful relations (all defined 'under the integral')

$$\delta(-x) = \delta(x)$$
$$x\delta(x) = 0$$
$$\delta(ax) = \frac{1}{|a|}\delta(x)$$
$$f(x)\delta(x-a) = f(a)\delta(x-a)$$
$$\int \delta(a-x)\delta(x-b)dx = \delta(a-b)$$
$$\delta(x^2-a^2) = \frac{\delta(x-a)+\delta(x+a)}{2|a|}$$
$$\delta[\varphi(x)] = \sum_i \frac{\delta(x-x_i)}{\left|\left(\frac{d\varphi}{dx}\right)_{x=x_i}\right|} \qquad (*)$$

where in (*) the x_i are the simple roots of the equation $\varphi(x) = 0$.

$\delta(x)$ is an even function of x, and it satisfies the equation:

$$\int_0^a \delta(x)dx = \begin{cases} \frac{1}{2}, \text{ if } a > 0; \\ -\frac{1}{2}, \text{ if } a < 0. \end{cases}$$

Limit of sequences of functions (defined under the integral)

$$\delta(x) = \lim_{a\to 0}\frac{1}{\pi}\frac{a}{a^2+x^2}$$

$$\delta(x) = \lim_{L\to\infty}\frac{\sin xL}{\pi x}$$

$$\delta(x) = \lim_{a\to 0}\frac{1}{\sqrt{2\pi a}}e^{-\frac{x^2}{2a}}$$

Further consequences

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk = \lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{+L} e^{ikx}dk = \lim_{L\to\infty}\frac{\sin xL}{\pi x} = \delta(x)$$

Separating the real and imaginary parts one finds:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\cos kx\ dk = \delta(x), \quad \int_{-\infty}^{+\infty}\sin kx\ dk = 0.$$

Further identities: $\delta'(x) = \frac{1}{\pi}\lim_{L\to\infty}\left[\frac{L\cos Lx}{x} - \frac{\sin Lx}{x^2}\right]$

$$\int \delta'(x)F(x)dx = -F'(0)$$

$$x\delta'(x) = -\delta(x)$$